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LIST EDGE COLOURINGS OF SOME 1-FACTORABLE MULTIGRAPHS

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The List Edge Colouring Conjecture asserts that, given any multigraph G with chromatic index k and any set system $\{S_e : e \in E(G)\}$ with each $|S_e| = k$, we can choose elements $s_e \in S_e$ such that $s_e \neq s_f$ whenever e and f are adjacent edges. Using a technique of Alon and Tarsi which involves the graph monomial $\prod\{xu - x_v : uv \in E\}$ of an oriented graph, we verify this conjecture for certain families of 1-factorable multigraphs, including 1-factorable planar graphs.

1 Introduction

Let $G = (V,E)$ be a graph (with multiple edges allowed). A proper (vertex) colouring of G is a function on V for which adjacent vertices receive distinct values. A *proper k-colouring* is a proper colouring whose range is a subset of $[k] := \{0,1,\ldots,k-1\}$. With this definition, two distinct proper k-colourings of G may induce the same partition of $V(G)$. A graph is k-colourable if it has a proper k-colouring. The following concept was introduced by Erdős, Rubin and Taylor [5]. Let $a : V(G) \to \{1,2,\ldots\}$. We say that G is *a-choosable* or *a-list colourable* if for every set system $\{S_v : v \in V\}$ such that $|S_v| = a(v)$, there is a proper colouring c such that $c(v) \in S_v$ for $v \in V(G)$. In case a is the constant function $a(v) \equiv k$, we say that G is *k-choosable*. The terms *k-edge colourable*, *a-edge choosable* and *k-edge choosable* are defined in an analogous way. If a graph is k-choosable then it is k-colourable, but not conversely, as shown by $K_{3,3}$ which is not 2-choosable. In contrast, we have the following.

Conjecture 1.1 (List Edge Colouring Conjecture). *If G is a k-edge colourable multigraph, then G is k-edge choosable.*

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This conjecture seems to have been arrived at independently by several people. It has been verified for the class of bipartite graphs [7], and also for complete graphs of odd order [8]. An excellent survey appears in [1]. Further results and historical comments may be found in [3, 4]. Our main result verifies this conjecture for a class of planar graphs.

Theorem 1.2. *If G is a d-regular d-edge colourable planar multigraph, then G is d-edge choosable.*

The Four Colour Theorem is equivalent to the statement that every 2-connected 3-regular planar graph is 3-edge colourable. Theorem 1.2 therefore implies that the Four Colour Theorem is equivalent to the statement that every 2-connected 3-regular planar graph is 3-edge choosable. This was observed independently by F. Jaeger and M. Tarsi (personal communication). For $d \geq 4$, the question of which d-regular planar multigraphs are d-edge colourable has not yet been resolved. Seymour [14] and others have proposed conjectures that would imply that any d-edge connected d-regular planar multigraph of even order is d-edge colourable, and hence, by Theorem 1.2, d-edge choosable.

Our main tool is a result of Alon and Tarsi [2] which relates choosability to coefficients in a certain polynomial. Let D be an orientation of G. The *graph monomial* of G is the homogeneous polynomial $\varepsilon(G)$ with variables $\{x_v : v \in V(G)\}$ and defined by

$$\varepsilon(G) = \prod_{uv \in E(D)} (x_u - x_v).$$

(Some authors call $\varepsilon(G)$ the *graph polynomial*, but we abandon this overused term in favour of that used by Sabidussi [12].) As we have defined it, $\varepsilon(G)$ depends on a particular orientation D of G; however changing the orientation multiplies $\varepsilon(G)$ by ± 1, so $\varepsilon(G)$ is unique up to sign. The graph monomial was first used by Petersen [11]; indeed Petersen gave *order*, *degree* and *factor* their graph theoretical meanings by reference to $\varepsilon(G)$. Scheim [13] used $\varepsilon(G)$ to prove some results about 3-edge colourings of 3-regular planar graphs; our Theorem 1.2 extends one of his results. Li and Li [10] mention $\varepsilon(G)$ in the context of determining the independence number of G.

Theorem 1.3 (Alon and Tarsi [2]). *Let $a : V(G) \to \{1,2,\ldots\}$. If the coefficient of $\prod_{v \in V(G)} x_v^{a(v)-1}$ in $\varepsilon(G)$ is nonzero, then G is a-choosable.*

Scheim's paper [13] contains much of the reasoning needed to prove this theorem; however, he was working before the introduction of the idea of list colourings, and did not state his results in full generality. Alon and Tarsi [2] give combinatorial interpretations of the coefficients of $\varepsilon(G)$, and use Theorem 1.3 to investigate the (vertex) choosability of planar graphs and bipartite graphs. Fleischner and Stiebitz [6] use Alon and Tarsi's results to solve a conjecture of Erdős regarding the 3-vertex colourability of certain 4-regular graphs. Penrose [17] states the case $d=3$ of Theorem 3.1 in terms of "abstract tensor systems".

2 Interpreting the coefficient

In order to study edge choosability one applies Theorem 1.3 to line graphs. The *line graph* $L(G)$ of a multigraph G has $V(L(G))=E(G)$ with an edge joining e to f in $L(G)$ for each common endpoint that e and f have in G. Thus, every pair of parallel edges in G is joined by *two* edges in $L(G)$. For regular G, the coefficient of $\varepsilon(L(G))$ which is of interest has several nice combinatorial interpretations, some of which are implicit in [2] and explicitly described by N. Alon in the preamble to Proposition 3.8 of [1].

From here on, G is a d-regular multigraph. Let $\xi(G)$ denote the coefficient of $\prod_{e\in E(G)} x_e^{d-1}$ in $\varepsilon(L(G))$. If $\xi(G)\neq 0$, then G is d-edge choosable, and thus the List Edge Colouring Conjecture holds true for G.

The set of edges $\delta(v)$ incident with each vertex v of G can be ordered with a *star labelling* at v, a bijection $\pi_v : \delta(v) \to [d]$. A *global star labelling* is a set $\pi = \{\pi_v : v \in V(G)\}$. We assume that G comes with a fixed global star labelling $\varrho = \varrho(G) = \{\varrho_v\}$, called the *reference labelling* of G, with which other star labellings will be compared. In particular, the *sign* of a star labelling π_v (relative to ϱ) is the sign of the permutation $\pi_v \circ \varrho_v^{-1}$, and is denoted $\operatorname{sign}_\varrho(\pi_v)$, or sometimes just $\operatorname{sign}(\pi_v)$. The sign of a global star labelling π is defined as $\operatorname{sign}(\pi)=\prod_{v\in V(G)} \operatorname{sign}(\pi_v)$.

Star labellings allow us to assign signs to other combinatorial objects in G. A *k-factor* in G is a k-regular spanning subgraph of G. Let $p=\lceil d/2\rceil$. An *ordered (near) 2-factorization* of G is an ordered partition $\boldsymbol{F}=(F_0,F_1,\ldots,F_{p-1})$ of $E(G)$, where each F_i is a 2-factor, unless d is odd, in which case F_{p-1} is a 1-factor (hence the word "near"). An *orientation* Φ of $\boldsymbol{F}$ is an orientation of G so that each F_i becomes a union Φ_i of directed circuits, except that when d is odd $\Phi_{p-1} = F_{p-1}$ remains an unoriented 1-factor. Let OOB2F(G) denote the set of oriented ordered (near) 2-factorizations of G in which each 2-factor is bipartite, i.e. a union of even circuits. For each $\Phi \in$ OOB2F(G), there is an associated global star labelling π: given $uv\in\Phi_i$ oriented from u to v, we set $\pi_u(uv)=i$ and $\pi_v(uv)=d-1-i$, or if d is odd and $uv\in\Phi_{p-1}$ then $\pi_u(uv)=\pi_v(uv)=(d-1)/2$. We define $\operatorname{sign}(\Phi)=\operatorname{sign}_\varrho(\Phi)$ to be $\operatorname{sign}(\pi)$. As shown in [1],

$$\xi(G) = \pm \sum_{\Phi\in \mathrm{OOB2F}(G)} \operatorname{sign}(\Phi). \tag{1}$$

Let B2F(G) denote the set of unordered and unoriented bipartite (near) 2-factorizations of G. For any $F \in$ B2F(G), we can define $\operatorname{sign}(F) = \operatorname{sign}_\varrho(F)$ to be $\operatorname{sign}(\Phi)$ for any orientation Φ of any ordering of F. All such Φ have the same sign, because reversing the orientation of an even circuit changes the sign at an even number of vertices, and swapping two 2-factors swaps two pairs of edges at each vertex. If $\omega(F)$ is the total number of circuits in all of the 2-factors in F, then there are $2^{\omega(F)}$ orientations of each of the $\lfloor d/2 \rfloor!$ orderings of F, so that (1) may be rewritten as

$$\xi(G) = \pm \lfloor d/2 \rfloor! \sum_{F \in \mathrm{B2F}(G)} \mathrm{sign}\,(F) 2^{\omega(F)}. \tag{2}$$

The coefficient $\xi(G)$ may also be interpreted in terms of edge colourings of G. Let $\mathrm{EC}_d(G)$ denote the set of proper d-edge colourings $c\colon E(G) \to [d]$. Each $c \in \mathrm{EC}_d(G)$ induces a global star labelling $\tau = \tau(c)$ where for each edge $e = uv$, $\tau_u(e) = \tau_v(e) = c(e)$. We define the sign of c (with respect to $\varrho(G)$) by $\mathrm{sign}(c) = \mathrm{sign}(\tau(c))$. As explained in [1], there is a bijection between $\mathrm{OOB2F}(G)$ and $\mathrm{EC}_d(G)$ which preserves all or reverses all signs, giving

$$\xi(G) = \pm \sum_{c \in \mathrm{EC}_d(G)} \mathrm{sign}\,(c). \tag{3}$$

Let $1\mathrm{F}(G)$ denote the set of unordered 1-factorizations of G. Each $f \in 1\mathrm{F}(G)$ corresponds to an equivalence class of $d!$ edge colourings in $\mathrm{EC}_d(G)$ under permutations of the colours $[d]$. As interchanging two colours in c introduces exactly $|V(G)|$ transpositions in $\tau(c)$, equivalent colourings in $\mathrm{EC}_d(G)$ have equal sign. Thus a sign function is well defined on $1\mathrm{F}(G)$.

$$\xi(G) = \pm d! \sum_{f \in 1\mathrm{F}(G)} \mathrm{sign}\,(f). \tag{4}$$

There is a coarser equivalence relation on $\mathrm{EC}_d(G)$ on whose parts a sign function can be defined. An *elementary Kempe recolouring* of $c \in \mathrm{EC}_d(G)$ exchanges the colours i and j on the edges of a single component circuit of the 2-factor $c^{-1}(i) \cup c^{-1}(j)$, for some distinct i, $j \in [d]$. Two elements of $\mathrm{EC}_d(G)$ (or $1\mathrm{F}(G)$) are *Kempe equivalent* if one can be obtained from the other by a sequence of elementary Kempe recolourings. Let $\mathrm{KE}(G)$ denote the set of *Kempe* (*equivalence*) *classes* of proper d-edge colourings of G. As with 1-factorizations, Kempe equivalent colourings have the same sign, and the sign of a Kempe class is well defined.

$$\xi(G) = \pm \sum_{\kappa \in \mathrm{KE}(G)} \mathrm{sign}\,(\kappa)|\kappa|. \tag{5}$$

We summarize with a list of sufficient conditions for a graph to be d-edge choosable.

Theorem 2.1. *Let G be a d-regular multigraph. Suppose that at least one of the following holds*

(i) *G has an odd number of distinct 1-factorizations,*

(ii) *G is 1-factorable and any two 1-factorizations are Kempe equivalent,*

(iii) *G is 1-factorable and any two 1-factorizations have the same sign, or*

(iv) *the number of* $F \in \mathrm{B2F}(G)$ *which minimize the total number* $\omega(F)$ *of circuits in all of the 2-factors in* F *is odd.*

Then $\xi(G) \neq 0$, *and as a consequence* G *is* d*-edge choosable.*

Proof. Claims (i) and (iii) follow immediately from (4), while (ii) follows from (5). If (iv) holds then the sum in (2) is non-zero modulo 2^{ω_0+1}, where $\omega_0 = \min\{\omega(F) : F \in \mathrm{B2F}(G)\}$. ∎

Note that condition (ii) implies condition (iii). We illustrate with some examples of d-regular graphs which are d-edge choosable by Theorem 2.1. The skeleton of the 3-cube has four distinct 1-factorizations, but they are all Kempe equivalent; thus (ii) applies, although (i) does not. The generalized Petersen graph $P(9,2)$ has a unique 1-factorization [15], and so (i) and (ii) both apply. Larger generalized Petersen graphs $P(6k+3,2)$, $k \geq 2$, are not uniquely 1-factorable, but have exactly three Hamilton circuits [15]. Thus $\omega(F)$ is minimum (equal to 1) for exactly three $F \in \mathrm{B2F}(G)$. These provide an example of (iv) whereas (i), (ii) and (iii) may not hold. The 8-vertex Möbius ladder (which may be thought of as an octagon with all four long diagonals added) has exactly three 1-factorizations, and they are all Kempe equivalent; therefore (i) and (ii) both apply. The skeleton of the dodecahedron has exactly ten 1-factorizations, each in its own Kempe class and all of the same sign; thus (iii) applies. The even complete graphs K_{2r} satisfy (iii) for $r \leq 3$, but not for $r \geq 4$. It appears likely that $\xi(K_{2r})$ is never zero (we have verified this electronically for $r \leq 5$), though this is probably a difficult problem. It is not even known whether the List Colouring Conjecture holds for K_{2r}. Similarly, we expect that $\xi(K_{2r,2r})$ is never zero (as has been verified for $r \leq 5$ by J. Janssen [private communication]), although (iii) holds only for $r \leq 2$.

In the next section we show that all 1-factorizations of a regular planar multigraph have the same sign. In contrast, $K_{3,3}$ has exactly one 1-factorization of each sign, thus $\xi(K_{3,3}) = 0$. (Even so, $K_{3,3}$ is 3-edge choosable as it is bipartite [7].) This is a special case of the situation for $K_{d,d}$ with $d \geq 3$ odd, which is discussed in [2]. More generally we have the following.

Proposition 2.2. *If* G *is* d*-regular, with* d *odd, and there exist distinct vertices* v, v' *with identical neighbourhoods, then* $\xi(G) = 0$.

Proof. We consider the involution on $\mathrm{EC}_d(G)$ which interchanges the colours of $v'w$ and $v'w$, for each neighbour w of v. This involution is fixed-point free and, as d is odd, is sign-reversing. Thus by (3), $\xi(G) = 0$. ∎

We briefly describe two operations which can be used to produce regular multigraphs G with $\xi(G) = 0$. Let G_0 and G_1 be disjoint d-regular multigraphs of even order, and let $v_i \in V(G_i)$ and $e_i \in E(G_i)$, $i = 0, 1$. We form a new d-regular multigraph H from $(G_0 - v_0) \cup (G_1 - v_1)$ by adding d new edges, each joining a neighbour of v_0 to a neighbour of v_1. We also form a new d-regular multigraph K from $(G_0 - e_0) \cup (G_1 - e_1)$ by adding two new edges, each joining an endpoint of e_0 to an endpoint of e_1. Using (3), one can show that $\xi(H) = \pm\xi(G_0)\xi(G_1)/d!$ and that $\xi(K) = \pm\xi(G_0)\xi(G_1)/d$. Thus $\xi(H) = \xi(K) = 0$ provided that $\xi(G_0) = 0$. Pavol Gvozdjak (personal communication) has found a Hamiltonian cubic graph G with $\xi(G) = 0$, but which does not arise from Proposition 2.2 nor either of these two operations. We do not know whether this graph is 3-edge choosable.

3. Regular planar multigraphs

In this section we prove Theorem 1.2 by showing the following

Theorem 3.1. *Let G be a d-regular planar multigraph, $d \geq 1$. Then all 1-factorizations of G have the same sign. Hence $|\xi(G)|$ is precisely the number of proper d-edge colourings of G.*

The case $d = 3$ of this theorem was proved by Scheim [13], and can also be deduced from a result of Vigneron [16] (see also Jaeger [9]) together with observations of Alon and Tarsi [2] relating the coefficients of $\varepsilon(G)$ to eulerian orientations of G. We leave as unsolved the problem of determining which graphs satisfy the conclusion of Theorem 3.1.

Roughly, we prove this theorem by giving a 'geometric' interpretation of $\operatorname{sign}(\Phi)$ in (1), and then using the topology of the plane to deduce that this sign is always positive. We use terminology and notation from Section 2. Let G be a d-regular graph embedded on an orientable surface. For $v \in V(G)$, a star labelling π is said to be *clockwise* if the edges are labelled in clockwise ascending order around v. A global star labelling $\pi = \{\pi_v\}$ of G is *clockwise* if each of its members is clockwise. From here on we assume the reference labelling $\varrho(G)$ to be clockwise. Let $\Phi = (\Phi_0, \ldots, \Phi_{p-1}) \in \mathrm{OOB2F}(G)$ and let v be a vertex of G. For $\Phi_i \in \Phi$ we denote by $\Phi_i(v)$ the connected component of Φ_i which contains v; thus $\Phi_i(v)$ is either an edge or a directed circuit. Two oriented 2-factors Φ_i, $\Phi_j \in \Phi$ are said to *cross* at v if the circuits $\Phi_i(v)$, $\Phi_j(v)$ geometrically cross at v. We say that an edge $e \in \delta(v) \setminus E(\Phi_i)$ lies *to the right* of Φ_i (at v) if e lies geometrically on the right as $\Phi_i(v)$ is traversed through v. Similarly, if v lies on the boundary of a face R of the embedding, then R is *to the left* of Φ_i (at v) if R lies geometrically on the left as $\Phi_i(c)$ is traversed through v. It is important to note that the terms 'cross' and 'to the left/right' can equally well (though more cumbersomely) be defined purely in terms of Φ and $\varrho(G)$, without reference to any embedding of G. For example, a face R is specified by a pair of edges in $\delta(v)$ having consecutive ϱ_v-labels (modulo d); two 2-factors Φ_i and Φ_j cross at v if some cyclic rotation of the sequence $\varrho_v \circ \pi_v^{-1}(i)$, $\varrho_v \circ \pi_v^{-1}(j)$, $\varrho_v \circ \pi_v^{-1}(d-1-i)$, $\varrho_v \circ \pi_v^{-1}(d-1-j)$ is monotone, where π is the global star labelling associated with Φ.

We define three invariants which determine the sign of Φ (relative to $\varrho(G)$). Let $v \in V(G)$. We denote by $x(\Phi, v)$ the number of unordered pairs of 2-factors in Φ which cross at v. If $d \geq 1$ is odd, then we define the *root edge* e_v to be the edge $\Phi_{p-1}(v)$; we let $r(\Phi, v)$ denote the number of oriented 2-factors $\Phi_i \in \Phi$ for which e_v lies to the right of Φ_i at v. If $d \geq 2$ is even, then we define the *root face* R_v to be the face specified by the ϱ_v-labels 0 and $d-1$; we let $l(\Phi, v)$ denote the number of oriented 2-factors $\Phi_i \in \Phi$ for which R_v lies to the left of Φ_i at v. Finally, we set $x(\Phi) := \sum_{v \in V(G)} x(\Phi, v)$, $r(\Phi) := \sum_{v \in V(G)} r(\Phi, v)$, and $l(\Phi) := \sum_{v \in V(G)} l(\Phi, v)$.

Lemma 3.2. *Let G be a d-regular multigraph with reference labelling ϱ. For any oriented ordered (near) 2-factorization Φ of G we have* $\operatorname{sign}(\Phi) = (-1)^{x(\Phi)+r(\Phi)}$ *or* $\operatorname{sign}(\Phi) = (-1)^{x(\Phi)+l(\Phi)}$ *according to whether d is odd or even.*

Proof. Given any star labelling π_v, let $\Phi(v)$ denote the oriented ordered partition of $\delta(v)$ whose ith part is the directed path with edges $\pi_v^{-1}(d-1-i)$ followed by $\pi_v^{-1}(i)$, except that when d is odd the $(p-1)$th part is the unoriented root edge $e_v = \pi_v^{-1}(p-1)$. In general, $x(\Phi, v)$ equals the number of pairs of paths in $\Phi(v)$ which cross, and $r(\Phi, v)$ ($l(\Phi, v)$) is the number of such paths for which e_v (R_v) lies to the right (left).

Let π be the global star labelling associated with Φ. For each v, $\Phi(v)$ is just the restriction of Φ to $\delta(v)$. A *ϱ-consecutive transposition* of π_v is any transposition which exchanges the π_v-labels on any two edges in $\delta(v)$ whose ϱ_v-labels differ by exactly one. The sign of π_v is determined by the length of a sequence S of such transpositions which transforms π_v into ϱ_v. In case d is odd, a ϱ-consecutive transposition of π_v corresponds to a modification of $\Phi(v)$ which does exactly one of two things. First, it may cross or uncross exactly one pair of dipaths in $\Phi(v)$. Second, it may transfer e_v from one side of exactly one such dipath to its other side. By definition, if $\pi_v = \varrho_v$, then $x(\Phi, v) = r(\Phi, v) = 0$. Thus $x(\Phi, v)+r(\Phi, v)$ is congruent to the number of transpositions in S (modulo 2), so $\operatorname{sign}(\pi_v) = (-1)^{x(\Phi,v)+r(\Phi,v)}$. Thus $\operatorname{sign}(\Phi) = \prod_{v \in V(G)} (-1)^{x(\Phi,v)+r(\Phi,v)} = (-1)^{x(\Phi)+r(\Phi)}$. The d-even case is exactly analogous, writing l and R_v in place of r and e_v. ∎

We remark here on an essential difference between the d-odd and d-even cases. The root edge e_v is determined by Φ whereas the root face R_v is defined by $\varrho(G)$. There appears to be no way of resolving this dichotomy.

A *plane* graph is a specific embedding of a planar graph in the plane. To prove Theorem 3.1 it suffices, by (1), to show that $x(\Phi)$, $r(\Phi)$ and $l(\Phi)$ are even, for any $\Phi \in \mathrm{OOB2F}(G)$, whenever G is plane and $\varrho(G)$ is clockwise. This (essentially) is proved in the next three lemmas.

Lemma 3.3. *Let G be a plane d-regular multigraph with a clockwise reference labelling ϱ. Then $x(\Phi)$ is even for any oriented ordered (near) 2-factorization Φ of G.*

Proof. Let x_{ij} denote the number of vertices at which two oriented 2-factors Φ_i, $\Phi_j \in \Phi$ cross. As any two edge-disjoint circuits in the plane geometrically cross an even number of times, each x_{ij} is even and thus $x(\Phi) = \sum_{ij} x_{ij}$ is even. ∎

In contrast to $x(\Phi)$, both $r(\Phi)$ and $l(\Phi)$ depend on the particular orientation Φ of the underlying (near) 2-factorization F. However, their parities are not affected by reorientation, provided that each of the 2-factors in F is bipartite. In case d is odd, we use the following simple observation whose proof is omitted

Proposition 3.4. *Let G be a plane 3-regular multigraph, and let C be a circuit of G. Let i be the number of vertices of C incident with an edge inside C, and j the number of vertices of G inside C. Then $i \equiv j \bmod 2$.*

Lemma 3.5. *Let G be a d-regular plane multigraph, where $d \geq 1$ is odd, and suppose that $\varrho(G)$ is clockwise. Then $r(\Phi)$ is even, for any oriented ordered bipartite near 2-factorization of G.*

Proof. For $0 \leq i \leq p-2$, let r_i be the number of vertices v for which the root edge e_v is to the right of the oriented 2-factor $\Phi_i \in \Phi$. As $r(\Phi) = \sum_{i=0}^{p-2} r_i$ it suffices to show that each r_i is even. For each i we argue as follows. We may assume that each circuit C in Φ_i is oriented clockwise so that 'to the right of Φ_i' is equivalent to 'inside C'. For any circuit C in Φ_i, the vertices of G inside C are the vertices of a union of even circuits in Φ_i. Thus an even number of vertices of G lie inside C. Applying Proposition 3.4 to the (undirected) 3-regular subgraph of G induced by the edges in $\Phi_i \cup \Phi_{p-1}$, there are an even number of vertices v in C for which e_v lies inside of C. As Φ_i is a disjoint union of circuits C, r_i is even as required. ∎

For the d-even case, we need some preliminary definitions. Let G be a plane graph, and C a circuit in G. We say that C *surrounds* a vertex v (or face R) if v (or R) is contained within the bounded region of $\mathbb{R}^2 - C$. If H is a 2-factor of G and v is a vertex of G, let $s(v,H)$ be the number of component circuits in H that surround v; define $s(R,H)$ similarly for a face R. Suppose G is a d-regular plane multigraph, where $d=2p$ is even. Then the plane dual of G is bipartite, and we can properly 2-face-colour G, using colours 0 and 1, so that the outer face is coloured 0. If G has a 2-factorization $F=\{F_0,F_1,\dots,F_{p-1}\}$, then it is not difficult to see that every face R receives the colour obtained by reducing modulo 2 the sum $s(R,F_0)+s(R,F_1)+\dots+s(R,F_{p-1})$. We say that the reference labelling $\varrho(G)$ is *0-consistent* if it is clockwise and each root face R_v, $v \in V(G)$ is coloured 0. We may assume that $\varrho(G)$ is 0-consistent.

Lemma 3.6. *Let G be a d-regular plane multigraph, where $d \geq 2$ is even, and suppose that $\varrho(G)$ is 0-consistent. Then $l(\Phi)$ is even, for any oriented ordered bipartite 2-factorization Φ of G.*

Proof. We may assume that each component circuit C in Φ_i is oriented anti-clockwise so that 'to the left of Φ_i' is equivalent to 'inside C'. For $0 \leq i \leq p-1$ and for each vertex v, let $l_i(v)$ equal 1 if R_v lies inside the circuit $\Phi_i(v)$, and 0 otherwise. For $v \in V(G)$ we consider the colour of R_v, which is the modulo 2 reduction of $\sum_{i=0}^{p-1} s(R_v,\Phi_i)$, and which is also 0, because ϱ is 0-consistent. For each i, $s(R_v,\Phi_i)=s(v,\Phi_i)+l_i(v)$. Therefore, working modulo 2, the colour of R_v is

$$0 \equiv \sum_{i=0}^{p-1} s(v,\Phi_i) + \sum_{i=0}^{p-1} l_i(v)$$

which implies

$$\sum_{i=0}^{p-1} l_i(v) \equiv \sum_{i=0}^{p-1} s(v,\Phi_i).$$

Therefore,

$$l(\Phi) = \sum_{v\in V(G)} \sum_{i=0}^{p-1} l_i(v) \equiv \sum_{i=0}^{p-1} \sum_{v\in V(G)} s(v,\Phi_i).$$

Since each component circuit C of each Φ_i has an even number of vertices, and $s(v,\Phi_i)$ is constant for all vertices of C, each sum $\sum_{v\in V(G)} s(v,\Phi_i)$ is even, and so $l(\Phi)$ is also even, as required. ∎

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References

[1] N. Alon: Restricted colorings of graphs, in: *"Surveys in Combinatorics", Proc. 14th British Combinatorial Conference, London Mathematical Society Lecture Notes Series* 187, edited by K. Walker, Cambridge University Press, 1993, 1–33.

[2] N. Alon, and M. Tarsi: Colorings and orientations of graphs, *Combinatorica*, **12** (1992), 125–134.

[3] B. Bollobás, and H. R. Hind: A new upper bound for the list chromatic number, *Discrete Math.*, **74** (1989), 65–75.

[4] Amanda Chetwynd, and Roland Häggkvist: A note on list-colorings, *J. Graph Theory*, **13** (1989), 87–95.

[5] P. Erdős, A. Rubin, and H. Taylor: Choosability in graphs, *Congr. Numer.*, **26** (1979), 125–157.

[6] H. Fleischner, and M. Stiebitz: A solution to a colouring problem of P. Erdős, *Discrete Math*, **101** (1992), 39–48.

[7] F. Galvin: The list chromatic index of a bipartite multigraph, *J. Combin. Theory, Ser. B*, **63** (1995), 153–159.

[8] R. Häggkvist, and J. Janssen: New bounds on the list-chromatic index of the complete graph, *Combin. Probab. Comput.*, to appear.

[9] F. Jaeger: On the Penrose number of cubic diagrams, *Discrete Math.*, **74** (1989), 85–97.

[10] S.-Y. R. Li, and W.-C. W. Li: Independence numbers of graphs and generators of ideals, *Combinatorica*, 1 (1981), 55–61.

[11] Julius Petersen: Die Theorie der regulären graphs, *Acta Math.*, **15** (1891), 193–220.

[12] G. Sabidussi: Binary invariants and orientations of graphs, *Discrete Math.*, **101** (1992), 251–277.

[13] David E. Scheim: The number of edge 3-colorings of a planar cubic graph as a permanent, *Discrete Math.*, **8** (1974), 377–382.

[14] P. D. Seymour: Some unsolved problems on one-factorizations of graphs, *Graph Theory and Related Topics*, edited by J. A. Bondy and U. S. R. Murty, Academic Press (1979) 367–368.

[15] Andrew Thomason: Cubic graphs with three hamiltonian cycles are not always uniquely edge colourable, *J. Graph Theory*, **6** (1982), 219–221.

[16] L. Vigneron: Remarques sur les réseaux cubiques de classe 3 associés au problème des quatre couleurs, *C. R. Acad. Sc. Paris*, **223** (1946), 770–772.

[17] Roger Penrose: Applications of negative dimensional tensors, in: *Combinatorial Mathematics and its Applications*, Proc. Conf., Oxford, 1969, Academic Press, London, 1971, 221–244.

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